BS1 test case
DNS of the Taylor-Green vortex at Re=1600

A. Nigro [1] , C. De Bartolo [1], F. Bassi [2]

[1] University of Calabria - Department of Mechanical, Energy and Management Engineering;
alessandra.nigro@unical.it, carmine.debartolo@unical.it
[2] University of Bergamo - Department of Engineering;
francesco.bassi@unibg.it

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Initial flow field

$$u = V_0 \sin\left(\frac{x}{L}\right)\cos\left(\frac{y}{L}\right)\cos\left(\frac{z}{L}\right)$$
$$v = -V_0 \cos\left(\frac{x}{L}\right)\sin\left(\frac{y}{L}\right)\cos\left(\frac{z}{L}\right)$$
$$w = 0$$
$$p = p_0 + \frac{\rho_0 V_0^2}{16}\left(\cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2y}{L}\right)\right)\left(\cos\left(\frac{2z}{L}\right) + 2\right)$$

where:

constant physical properties
$Re = 1600$
$M_0 = 0.1$
$\gamma = 1.4$
$Pr = 0.71$

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MIGALE code:

- DGFEM
- Compressible NS equations
- Primitive variables with $p = \ln p$ and $T = \ln T$

Taylor-Green vortex

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Discretization

- $x, y, z \in [-\pi, \pi]$
- Periodic boundary conditions
- $T = 20t_c$
- Total DOFs $\forall$ unknown=256^3, 128^3, 64^3
- Uniform cartesian grid
- $P1 - P5$ elements
- ROS(5,8) vs. RK(4,5) time integration schemes
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By considering uniform cartesian grids:

		256^3 dofs		128^3 dofs		64^3 dofs	
P	dofs ∀el.	n. el.	el.∀ dir.	n. el.	el.∀ dir.	n. el.	el.∀ dir.
1	4	4,194,304	161	–		–	–
2	10	1,677,722	119	209,715	59	–	–
3	20	838,861	94	104,858	47	13,107	24
4	35	–	–	59,919	39	7,490	20
5	56	–	–	–	–	4,681	17

Outputs:

- Temporal evolution of the kinetic energy integrated on the domain:

$$E_k = \frac{1}{\rho_0 \Omega} \int_{\Omega} \rho \frac{\mathbf{v} \cdot \mathbf{v}}{2} d\Omega$$

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$$\epsilon = -\frac{dE_k}{dt}$$

- Temporal evolution of the dissipation rate ϵ based on the enstrophy ε integrated on the domain:

$$\epsilon = 2\frac{\mu}{\rho_0}\varepsilon$$

where

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- Temporal evolution of the theoretical error defined as:

$$\text{Theoretical error} = \left| \frac{dE_k}{dt} + \epsilon \right|$$

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Analysis:

- Evolution of the outputs as a function of the time compared with the results of a pseudo-spectral code
- Evolution of the outputs errors with respect to the results of a pseudo-spectral code as a function of the time
- Maximum outputs errors vs. W.U.

Outputs Errors:

- Temporal evolution of the kinetic energy errors with respect to the results of a pseudo-spectral code:

$$\Delta E_k = |E_k - E_k^*|_{t\in[0,20t_c]}$$

$$\Delta E_k\% = \left|\frac{E_k - E_k^*}{E_k^*}\right|_{t\in[0,20t_c]} \times 100$$

- Temporal evolution of the kinetic energy dissipation rate errors with respect to the results of a pseudo-spectral code:

$$\Delta\frac{dE_k}{dt} = \left|-\frac{dE_k}{dt} + \left(\frac{dE_k}{dt}\right)^*\right|_{t\in[0,20t_c]}$$

$$\Delta\frac{dE_k}{dt}\% = \left|\frac{-\frac{dE_k}{dt} + \left(\frac{dE_k}{dt}\right)^*}{\left(\frac{dE_k}{dt}\right)^*}\right|_{t\in[0,20t_c]} \times 100$$

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$$\Delta\epsilon = |\epsilon - \epsilon^*|_{t\in[0,20t_c]}$$

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Maximum Outputs Errors:

- Maximum kinetic energy errors with respect to the results of a pseudo-spectral code:

$$\text{err}\left(E_k\right) = \left[\max\left(\Delta E_k\right)\right]_{t\in[0,10t_c]}$$

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- Maximum kinetic energy dissipation rate errors with respect to the results of a pseudo-spectral code:

$$\text{err}\left(\frac{dE_k}{dt}\right) = \left[\max\left(\Delta\frac{dE_k}{dt}\right)\right]_{t\in[0,10t_c]}$$

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- Maximum dissipation rate ϵ errors with respect to the results of a pseudo-spectral code:

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Computational resources employed:

- In-house resources at the High Performance Computing Center (HPCC) of University of Calabria: 10 computing nodes with 20 cores each (Intel(R) Xeon(R) CPU E5-2680 v2 @ 2.80GHz)
- Simulation are performed on $40 - 200$ cores

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Explicit Runge Kutta(4,5) & ERS numerical flux

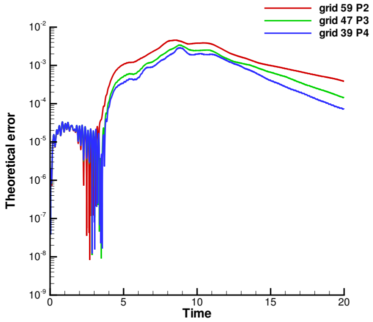
		256^3 dofs			128^3 dofs			64^3 dofs		
P	cfl	grid	$T/\Delta t_{av}$	W.U.	grid	$T/\Delta t_{av}$	W.U.	grid	$T/\Delta t_{av}$	W.U.
1	0.35	161	16,443	$3.196 \cdot 10^6$	–	–	–	–	–	–
2	0.2	119	21,149	$6.954 \cdot 10^6$	59	10,310	$3.892 \cdot 10^5$	–	–	–
3	0.15	94	22,099	$1.515 \cdot 10^7$	47	10,907	$9.100 \cdot 10^5$	24	5,443	$6.374 \cdot 10^4$
4	0.1	–	–	–	39	13,527	$2.745 \cdot 10^6$	20	6,770	$1.874 \cdot 10^5$
5	0.1	–	–	–	–	–	–	17	5,783	$3.856 \cdot 10^5$

Variable Δt with $cfl = 1/(2P+1)$

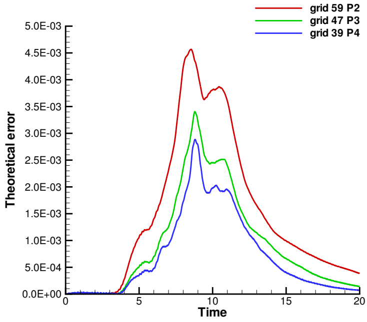
Tau Bench $=7.208$ s

128^3 DOFs

Theoretical error log scale:



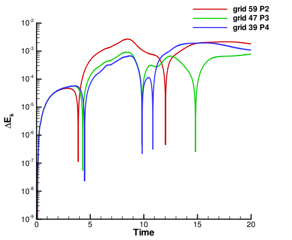
Theoretical error non-log scale:



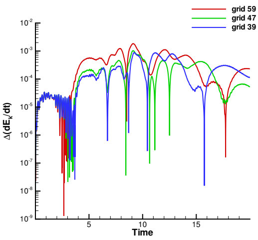
Polynomial degree

128^3 DOFs

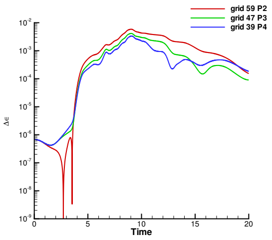
ΔE_k:



$\Delta\left(dE_k/dt\right)$:



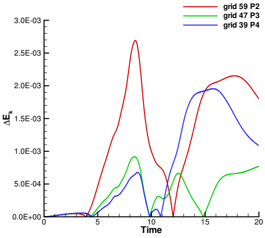
$\Delta\epsilon$:



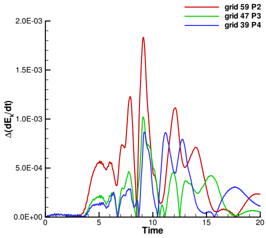
Polynomial degree

128^3 DOFs

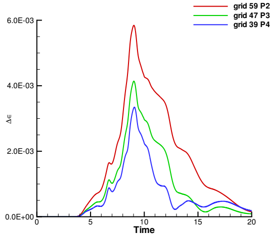
ΔE_k:



$\Delta\left(dE_k/dt\right)$:

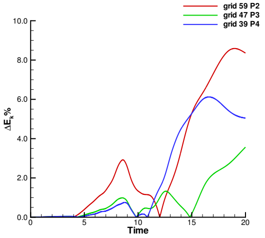


$\Delta\epsilon$:

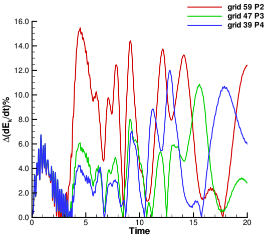


128^3 DOFs

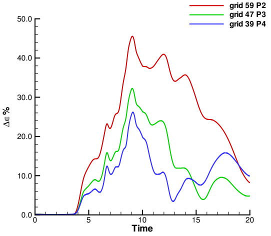
$\Delta E_k\%$:



$\Delta\left(dE_k/dt\right)\%$:



$\Delta\epsilon\%$:



max errors RK(4,5) & ERS numerical flux:

$$\text{err}(a) = \max\left(|a - a^*|_{t\in[0,10t_c]}\right)$$
$$\text{err}(a)\,\% = \max\left([|a - a^*/a^*|]_{t\in[0,10t_c]}\right) \times 100$$

256^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
1	161	$5.099 \cdot 10^{-3}$	5.61	$1.587 \cdot 10^{-3}$	34.22	$6.667 \cdot 10^{-3}$	51.97	$5.836 \cdot 10^{-3}$
2	119	$3.232 \cdot 10^{-4}$	0.36	$4.153 \cdot 10^{-4}$	6.74	$2.196 \cdot 10^{-3}$	17.22	$1.995 \cdot 10^{-3}$
3	94	$1.122 \cdot 10^{-4}$	0.13	$2.618 \cdot 10^{-4}$	6.74	$1.098 \cdot 10^{-3}$	8.73	$9.781 \cdot 10^{-4}$

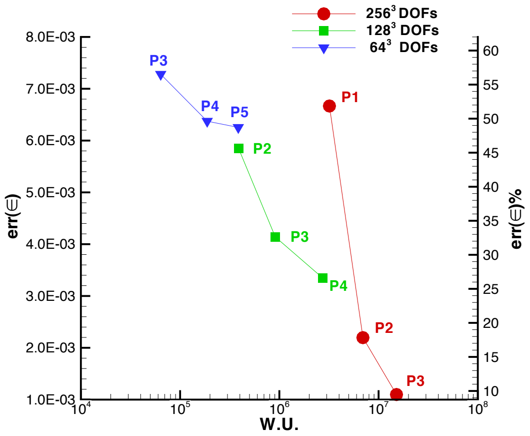
128^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
2	59	$2.692 \cdot 10^{-3}$	2.92	$1.833 \cdot 10^{-3}$	15.47	$5.844 \cdot 10^{-3}$	45.54	$4.566 \cdot 10^{-3}$
3	47	$9.161 \cdot 10^{-4}$	0.99	$1.020 \cdot 10^{-3}$	7.99	$4.140 \cdot 10^{-3}$	32.25	$3.404 \cdot 10^{-3}$
4	39	$6.777 \cdot 10^{-4}$	0.75	$8.659 \cdot 10^{-4}$	7.02	$3.345 \cdot 10^{-3}$	26.19	$2.885 \cdot 10^{-3}$

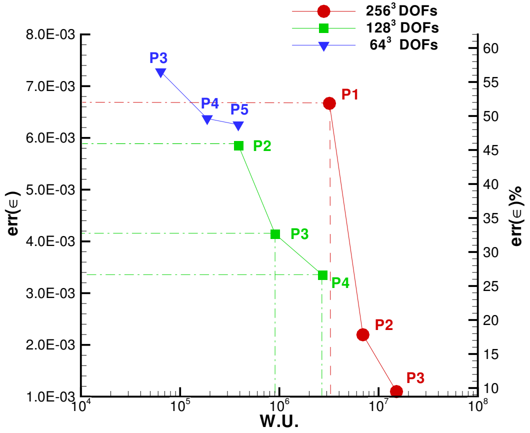
64^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
3	24	$6.847 \cdot 10^{-3}$	7.45	$2.436 \cdot 10^{-3}$	36.58	$7.280 \cdot 10^{-3}$	56.82	$7.318 \cdot 10^{-3}$
4	20	$4.240 \cdot 10^{-3}$	4.92	$2.266 \cdot 10^{-3}$	21.19	$6.374 \cdot 10^{-3}$	49.68	$6.893 \cdot 10^{-3}$
5	17	$3.694 \cdot 10^{-3}$	4.07	$2.231 \cdot 10^{-3}$	21.73	$6.254 \cdot 10^{-3}$	48.72	$5.634 \cdot 10^{-3}$

error(ϵ)/W.U. RK(4,5) & ERS numerical flux:



error(ϵ)/W.U. RK(4,5) & ERS numerical flux:



Implicit Rosenbrock(5,8) vs. Explicit Runge Kutta(4,5)

ROS(5,8)

	256^3 dofs					128^3 dofs					64^3 dofs				
P	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)
1	161	0.12	167	1.174	272	–	–	–	–	–	–	–	–	–	–
2	119	0.1	200	2.934	608	59	0.17	119	2.476	76	–	–	–	–	–
3	94	0.1	200	2.057	1,140	47	0.12	167	2.447	144	24	0.17	119	1.890	23
4	–	–	–	–	–	39	0.12	167	3.222	248	20	0.17	119	2.372	36
5	–	–	–	–	–	–	–	–	–	–	17	0.17	119	2.292	56

Implicit Rosenbrock(5,8) vs. Explicit Runge Kutta(4,5)

ROS(5,8)

	256^3 dofs					128^3 dofs					64^3 dofs				
P	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)
1	161	0.12	167	1.174	272	–	–	–	–	–	–	–	–	–	–
2	119	0.1	200	2.934	608	59	0.17	119	2.476	76	–	–	–	–	–
3	94	0.1	200	2.057	1,140	47	0.12	167	2.447	144	24	0.17	119	1.890	23
4	–	–	–	–	–	39	0.12	167	3.222	248	20	0.17	119	2.372	36
5	–	–	–	–	–	–	–	–	–	–	17	0.17	119	2.292	56

RK(4,5)

		256^3 dofs				128^3 dofs				64^3 dofs			
P	cfl	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)
1	0.35	161	16,443	$3.196 \cdot 10^6$	28	–	–	–	–	–	–	–	–
2	0.2	119	21,149	$6.954 \cdot 10^6$	88	59	10,310	$3.892 \cdot 10^5$	12	–	–	–	–
3	0.15	94	22,099	$1.515 \cdot 10^7$	131	47	10,907	$9.100 \cdot 10^5$	18	24	5,443	$6.374 \cdot 10^4$	5
4	0.1	–	–	–	–	39	13,527	$2.745 \cdot 10^6$	27	20	6,770	$1.874 \cdot 10^5$	6
5	0.1	–	–	–	–	–	–	–	–	17	5,783	$3.856 \cdot 10^5$	8

Implicit Rosenbrock(5,8) vs. Explicit Runge Kutta(4,5)

ROS(5,8)

	256^3 dofs					128^3 dofs					64^3 dofs				
P	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)
1	161	0.12	167	1.174	272	–	–	–	–	–	–	–	–	–	–
2	119	0.1	200	2.934	608	59	0.17	119	2.476	76	–	–	–	–	–
3	94	0.1	200	2.057	1,140	47	0.12	167	2.447	144	24	0.17	119	1.890	23
4	–	–	–	–	–	39	0.12	167	3.222	248	20	0.17	119	2.372	36
5	–	–	–	–	–	–	–	–	–	–	17	0.17	119	2.292	56

RK(4,5)

		256^3 dofs				128^3 dofs				64^3 dofs			
P	cfl	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)
1	0.35	161	16,443	$3.196 \cdot 10^6$	28	–	–	–	–	–	–	–	–
2	0.2	119	21,149	$6.954 \cdot 10^6$	88	59	10,310	$3.892 \cdot 10^5$	12	–	–	–	–
3	0.15	94	22,099	$1.515 \cdot 10^7$	131	47	10,907	$9.100 \cdot 10^5$	18	24	5,443	$6.374 \cdot 10^4$	5
4	0.1	–	–	–	–	39	13,527	$2.745 \cdot 10^6$	27	20	6,770	$1.874 \cdot 10^5$	6
5	0.1	–	–	–	–	–	–	–	–	17	5,783	$3.856 \cdot 10^5$	8

- W.U.$_{ratio}$ =W.U.$_{RK(4,5)}$/W.U.$_{ROS(5,8)}$ $= 2 - 3$
- RAM$_{ratio}$ =RAM$_{ROS(5,8)}$/RAM$_{RK(4,5)}$ $= 6 - 10$

Implicit Rosenbrock(5,8) vs. Explicit Runge Kutta(4,5)

ROS(5,8)

	256^3 dofs					128^3 dofs					64^3 dofs				
P	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)
1	161	0.12	167	1.174	272	–	–	–	–	–	–	–	–	–	–
2	119	0.1	200	2.934	608	59	0.17	119	2.476	76	–	–	–	–	–
3	94	0.1	200	2.057	1,140	47	0.12	167	2.447	144	24	0.17	119	1.890	23
4	–	–	–	–	–	39	0.12	167	3.222	248	20	0.17	119	2.372	36
5	–	–	–	–	–	–	–	–	–	–	17	0.17	119	2.292	56

RK(4,5)

		256^3 dofs				128^3 dofs				64^3 dofs			
P	cfl	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	grid	$T/\Delta t_{av}$	W.U.	RAM (GB)
1	0.35	161	16,443	$3.196 \cdot 10^6$	28	–	–	–	–	–	–	–	–
2	0.2	119	21,149	$6.954 \cdot 10^6$	88	59	10,310	$3.892 \cdot 10^5$	12	–	–	–	–
3	0.15	94	22,099	$1.515 \cdot 10^7$	131	47	10,907	$9.100 \cdot 10^5$	18	24	5,443	$6.374 \cdot 10^4$	5
4	0.1	–	–	–	–	39	13,527	$2.745 \cdot 10^6$	27	20	6,770	$1.874 \cdot 10^5$	6
5	0.1	–	–	–	–	–	–	–	–	17	5,783	$3.856 \cdot 10^5$	8

- W.U.$_{ratio}$ =W.U.$_{RK(4,5)}$/W.U.$_{ROS(5,8)}$ $= 2 - 3$
- RAM$_{ratio}$ =RAM$_{ROS(5,8)}$/RAM$_{RK(4,5)}$ $= 6 - 10$

Implicit Rosenbrock(5,8) vs. Explicit Runge Kutta(4,5)

ROS(5,8)

P	256^3 dofs grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	128^3 dofs grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)	64^3 dofs grid	f	$T/\Delta t$	W.U.$_{ratio}$	RAM (GB)
1	161	0.12	167	1.174	272	–	–	–	–	–	–	–	–	–	–
2	119	0.1	200	2.934	608	59	0.17	119	2.476	76	–	–	–	–	–
3	94	0.1	200	2.057	1,140	47	0.12	167	2.447	144	24	0.17	119	1.890	23
4	–	–	–	–	–	39	0.12	167	3.222	248	20	0.17	119	2.372	36
5	–	–	–	–	–	–	–	–	–	–	17	0.17	119	2.292	56

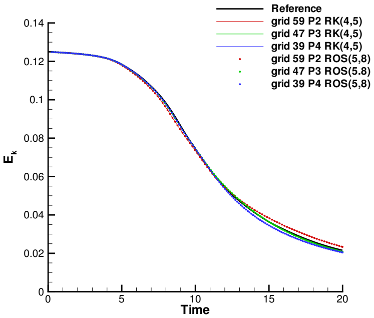
RK(4,5)

P	cfl	256^3 dofs grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	128^3 dofs grid	$T/\Delta t_{av}$	W.U.	RAM (GB)	64^3 dofs grid	$T/\Delta t_{av}$	W.U.	RAM (GB)
1	0.35	161	16,443	$3.196 \cdot 10^6$	28	–	–	–	–	–	–	–	–
2	0.2	119	21,149	$6.954 \cdot 10^6$	88	59	10,310	$3.892 \cdot 10^5$	12	–	–	–	–
3	0.15	94	22,099	$1.515 \cdot 10^7$	131	47	10,907	$9.100 \cdot 10^5$	18	24	5,443	$6.374 \cdot 10^4$	5
4	0.1	–	–	–	–	39	13,527	$2.745 \cdot 10^6$	27	20	6,770	$1.874 \cdot 10^5$	6
5	0.1	–	–	–	–	–	–	–	–	17	5,783	$3.856 \cdot 10^5$	8

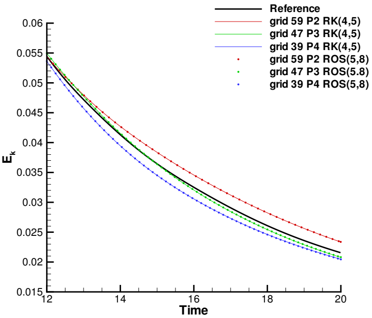
- W.U.$_{ratio}$ = W.U.$_{RK(4,5)}$/W.U.$_{ROS(5,8)}$ $= 2 - 3$
- RAM$_{ratio}$ = RAM$_{ROS(5,8)}$/RAM$_{RK(4,5)}$ $= 6 - 10$

128^3 DOFs

E_k:

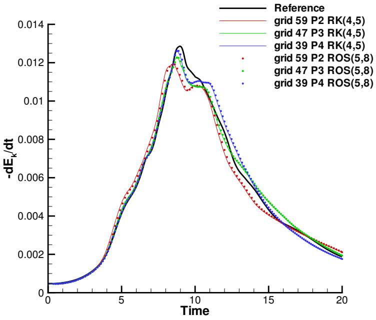


Zoom E_k:

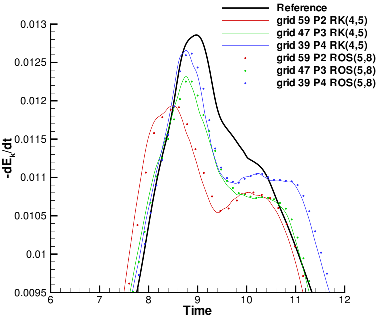


128^3 DOFs

$-dE_k/dt$:

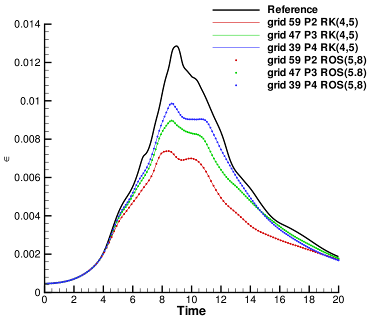


Zoom $-dE_k/dt$:

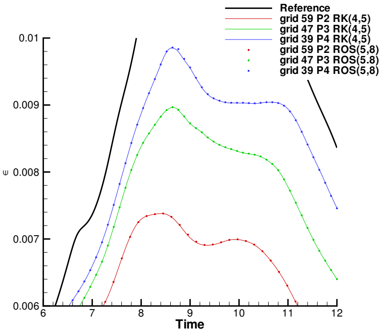


128^3 DOFs

ϵ:

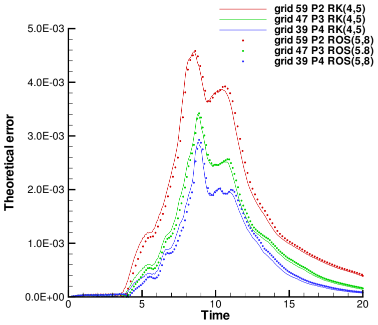


Zoom ϵ:

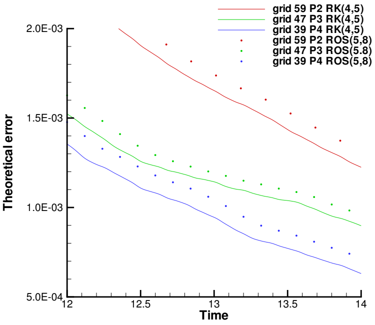


128^3 DOFs

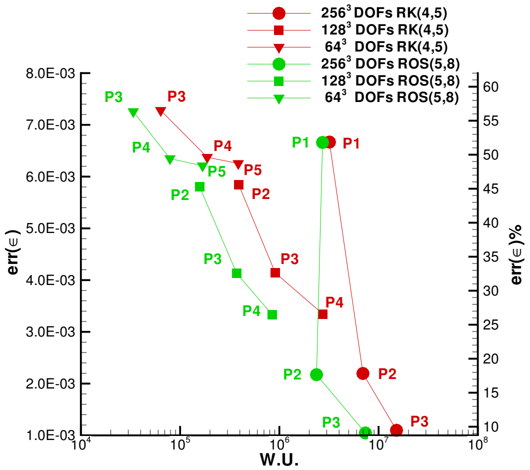
Theoretical error:



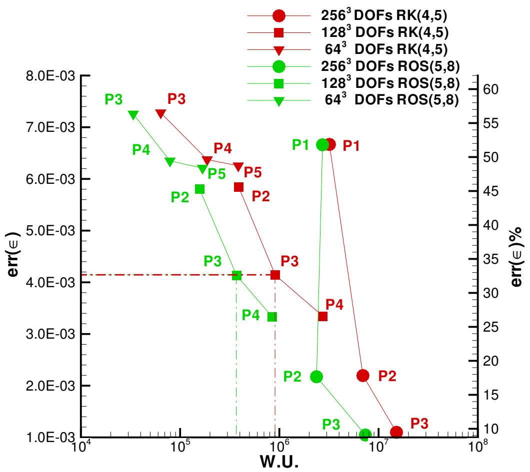
Zoom theoretical error:



error(ϵ)/W.U. ROS(5,8) vs. RK(4,5):



error(ϵ)/W.U. ROS(5,8) vs. RK(4,5):



Implicit Rosenbrock(5,8): Preconditioned Roe vs. ERS

ROS(5,8) ERS:

	256^3 dofs			128^3 dofs			64^3 dofs		
P	grid	$T/\Delta t$	W.U.	grid	$T/\Delta t$	W.U.	grid	$T/\Delta t$	W.U.
1	161	167	$2.723 \cdot 10^6$	–	–	–	–	–	–
2	119	200	$2.370 \cdot 10^6$	59	119	$1.572 \cdot 10^5$	–	–	–
3	94	200	$7.365 \cdot 10^6$	47	167	$3.719 \cdot 10^5$	24	119	$3.372 \cdot 10^4$
4	–	–	–	39	167	$8.522 \cdot 10^5$	20	119	$7.901 \cdot 10^4$
5	–	–	–	–	–	–	17	119	$1.683 \cdot 10^5$

ROS(5,8) Preconditioned Roe:

	256^3 dofs			128^3 dofs			64^3 dofs		
P	grid	$T/\Delta t$	W.U.$_{ratio}$	grid	$T/\Delta t$	W.U.$_{ratio}$	grid	$T/\Delta t$	W.U.$_{ratio}$
1	161	250	1.112	–	–	–	–	–	–
2	119	250	1.205	59	167	1.337	–	–	–
3	94	334	1.588	47	200	1.261	24	119	1.077
4	–	–	–	39	200	1.263	20	119	1.057
5	–	–	–	–	–	–	17	119	1.064

$$\text{W.U.}_{ratio} = \text{W.U.}_{p-Roe} / \text{W.U.}_{ERS}$$

Implicit Rosenbrock(5,8): Preconditioned Roe vs. ERS

ROS(5,8) ERS:

	256^3 dofs			128^3 dofs			64^3 dofs		
P	grid	$T/\Delta t$	W.U.	grid	$T/\Delta t$	W.U.	grid	$T/\Delta t$	W.U.
1	161	167	$2.723 \cdot 10^6$	–	–	–	–	–	–
2	119	200	$2.370 \cdot 10^6$	59	119	$1.572 \cdot 10^5$	–	–	–
3	94	200	$7.365 \cdot 10^6$	47	167	$3.719 \cdot 10^5$	24	119	$3.372 \cdot 10^4$
4	–	–	–	39	167	$8.522 \cdot 10^5$	20	119	$7.901 \cdot 10^4$
5	–	–	–	–	–	–	17	119	$1.683 \cdot 10^5$

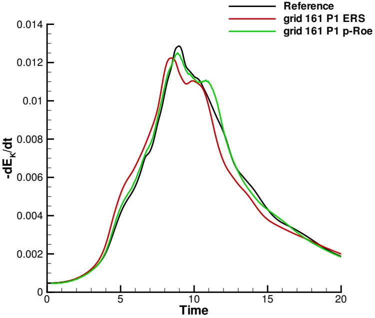
ROS(5,8) Preconditioned Roe:

	256^3 dofs			128^3 dofs			64^3 dofs		
P	grid	$T/\Delta t$	W.U.$_{ratio}$	grid	$T/\Delta t$	W.U.$_{ratio}$	grid	$T/\Delta t$	W.U.$_{ratio}$
1	161	250	1.112	–	–	–	–	–	–
2	119	250	1.205	59	167	1.337	–	–	–
3	94	334	1.588	47	200	1.261	24	119	1.077
4	–	–	–	39	200	1.263	20	119	1.057
5	–	–	–	–	–	–	17	119	1.064

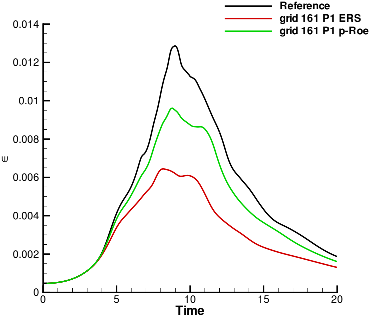
W.U.$_{ratio}$ =W.U.$_{p-Roe}$/W.U.$_{ERS}$

256^3 DOFs $P1$

E_k dissipation rate:

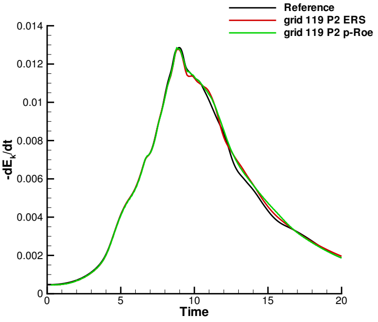


E_k dissipation rate based on ε:

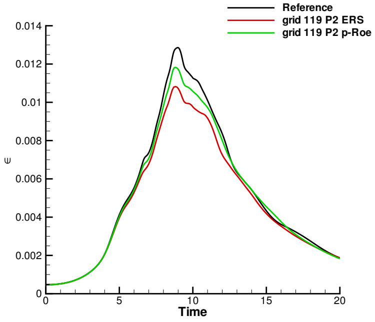


256^3 DOFs $P2$

E_k dissipation rate:

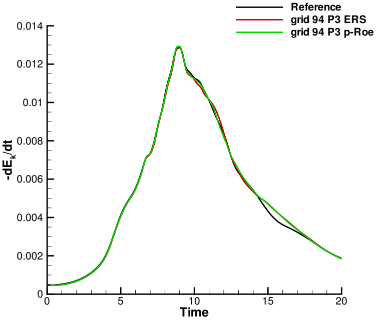


E_k dissipation rate based on ε:

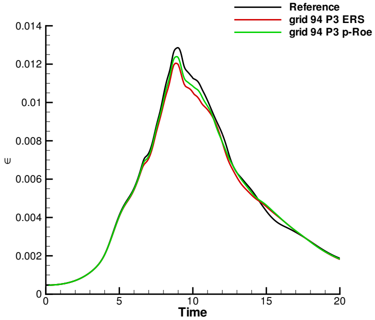


256^3 DOFs $P3$

E_k dissipation rate:

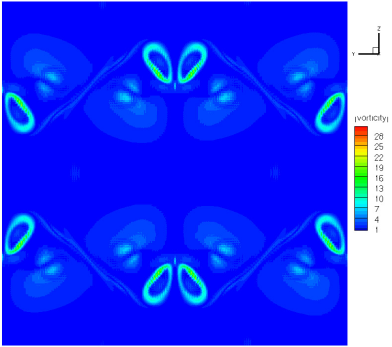


E_k dissipation rate based on ε:

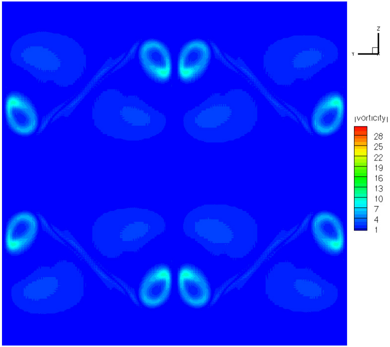


256^3 DOFs $P1$: vorticity norm at $x = -\pi L$ at time $t/t_c = 8$

p-Roe:

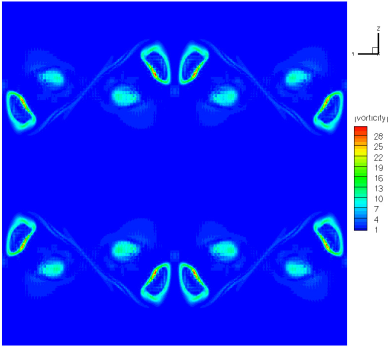


ERS:

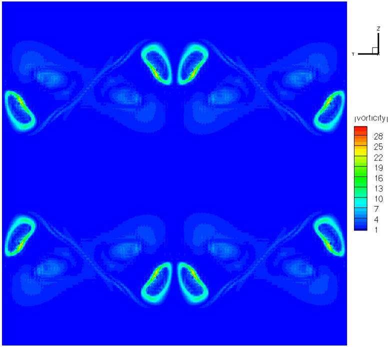


256^3 DOFs $P2$: vorticity norm at $x = -\pi L$ at time $t/t_c = 8$

p-Roe:

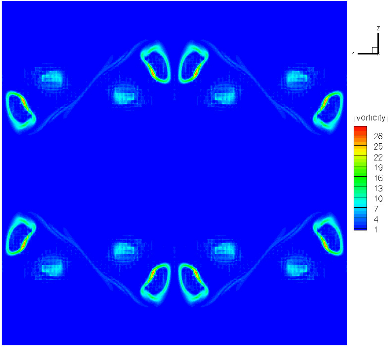


ERS:

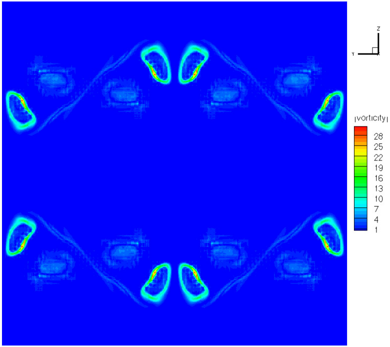


256^3 DOFs $P3$: vorticity norm at $x = -\pi L$ at time $t/t_c = 8$

p-Roe:



ERS:



max errors Preconditioned Roe:

$$\text{err}(a) = \max\left(|a - a^*|_{t\in[0,10t_c]}\right)$$
$$\text{err}(a)\,\% = \max\left([|a - a^*|\,/a^*]_{t\in[0,10t_c]}\right) \times 100$$

256^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
1	161	$1.670 \cdot 10^{-3}$	1.83	$5.482 \cdot 10^{-4}$	8.60	$3.367 \cdot 10^{-3}$	26.23	$2.929 \cdot 10^{-3}$
2	119	$1.044 \cdot 10^{-4}$	0.12	$1.964 \cdot 10^{-4}$	8.11	$1.185 \cdot 10^{-3}$	9.35	$1.047 \cdot 10^{-3}$
3	94	$1.176 \cdot 10^{-4}$	0.14	$1.259 \cdot 10^{-4}$	7.00	$6.807 \cdot 10^{-4}$	5.48	$6.381 \cdot 10^{-4}$

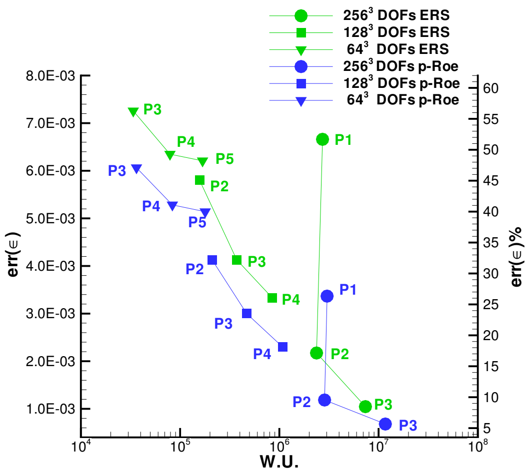
128^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
2	59	$1.196 \cdot 10^{-3}$	1.30	$8.351 \cdot 10^{-4}$	7.54	$4.128 \cdot 10^{-3}$	32.29	$3.359 \cdot 10^{-3}$
3	47	$6.205 \cdot 10^{-4}$	0.70	$4.623 \cdot 10^{-4}$	6.81	$2.999 \cdot 10^{-3}$	23.49	$2.735 \cdot 10^{-3}$
4	39	$4.820 \cdot 10^{-4}$	0.56	$1.873 \cdot 10^{-4}$	7.65	$2.301 \cdot 10^{-3}$	18.02	$2.261 \cdot 10^{-3}$

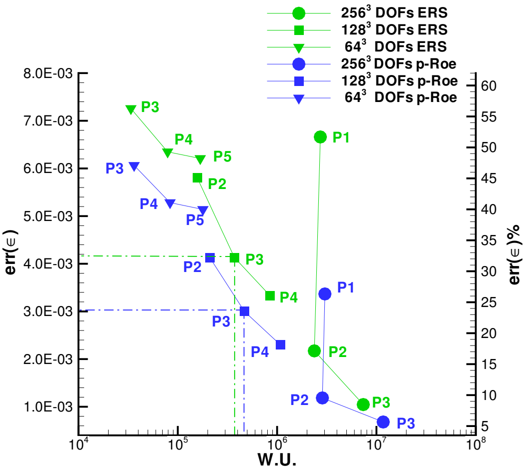
64^3 DOFs

P	grid	err(E_k)	err(E_k) %	err(dE_k/dt)	err(dE_k/dt)%	err(ϵ)	err(ϵ)%	Theor. err.
3	24	$5.939 \cdot 10^{-3}$	7.29	$2.042 \cdot 10^{-3}$	24.19	$6.064 \cdot 10^{-3}$	47.53	$6.125 \cdot 10^{-3}$
4	20	$3.739 \cdot 10^{-3}$	4.90	$1.781 \cdot 10^{-3}$	17.46	$5.283 \cdot 10^{-3}$	41.10	$5.138 \cdot 10^{-3}$
5	17	$3.633 \cdot 10^{-3}$	3.63	$1.506 \cdot 10^{-3}$	13.98	$5.140 \cdot 10^{-3}$	40.35	$4.821 \cdot 10^{-3}$

error(ϵ)/W.U. ROS(5,8): ERS vs. Preconditioned Roe



error(ϵ)/W.U. ROS(5,8): ERS vs. Preconditioned Roe



Conclusions

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- Performance of a MF-MEBDF-DG scheme

Biblio:

- F. Bassi, L. Botti, A. Colombo, A. Ghidoni and F. Massa. Linearly implicit Rosenbrock-type Runge-Kutta schemes applied to the Discontinuous Galerkin solution of compressible and incompressible unsteady flows. Computer & Fluids, Vol. 118, pp. 305–320, (2015)
- F. Bassi , C. De Bartolo , R. Hartmann and A. Nigro. A discontinuous Galerkin method for inviscid low Mach number flows. Journal of Computational Physics, 228(11), pp. 3996–4011, (2009)
- A. Nigro, S. Renda, C. De Bartolo, R. Hartmann and F. Bassi. A high-order accurate discontinuous Galerkin finite element method for laminar low Mach number flows. International Journal for Numerical Methods in Fluids, 72(1), pp.43–68, (2013)

That's all

Thank you