Computation the bow shock at M=4 with a 5th-order WENO scheme

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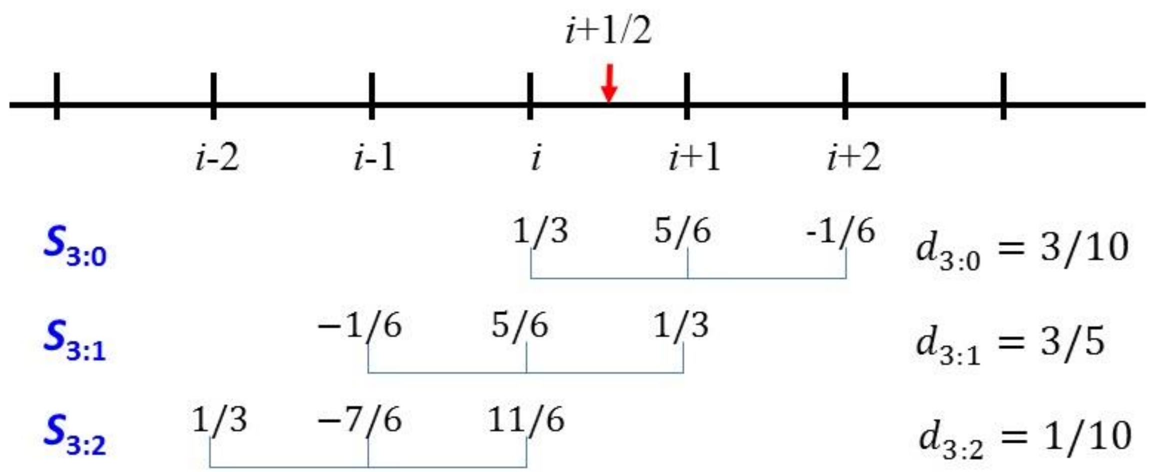
Spatial discretization: WENO-5

- Weighted average of interpolated values from all candidate stencils

$$\tilde{v}^{r}_{i+1/2} = \sum_{j=0}^{k-1} c_{rj} v_j$$

$$v_{i+1/2} = \sum_{r=0}^{k-1} \omega_k \tilde{v}^{r}_{i+1/2}$$



WENO-5

Shu, 2000, NASA/CR-97-206253
Henric et. al. 2005, *JCP*
Bogers et. al. 2008, *JCP*

Implicit time integration

- Semi-discretized form

$$\frac{\partial Q}{\partial t} = R(Q)$$

- Implicit time integration
  - Significantly relaxed stability constraints
  - Large time steps (limited by physical considerations), reduce the required CPU time
  - Main difficulty solution of the resulting nonlinear system of equations

Implicit time integration - 1

- Implicit1st order backward Euler

$$\frac{Q^{n+1} - Q^{n}}{\Delta t} = R(Q^{n+1})$$

Nonlinear implicit equations

$$F(Q^{n+1}) = \frac{1}{\Delta t} Q^{n+1} - R(Q^{n+1}) - \frac{Q^{n}}{\Delta t} = 0$$

Implicit time integration - 2

- S-stage, implicit Runge-Kutta schemes

$$Y_i = Q^n + \Delta t \sum_{j=1}^{i} a_{ij} R\left(t_n + c_j \Delta t, Y_j\right) \qquad i = 1, 2, \ldots, s$$

$$Q^{n+1} = Q^n + \Delta t \sum_{i=1}^{s} b_i R(t_n + c_i \Delta t\,, Y_i)$$

i-stage

$$F(Y_i) = \frac{1}{\Delta t} Y_i - a_{ii} R(t_n + c_i \Delta t, Y_i) - \left[\frac{Q^n}{\Delta t} + \sum_{j=1}^{i-1} a_{ij} R(t_n + c_j \Delta t, Y_j \right] = 0$$

Solution of implicit equations

- General equations

$$F(Y_i) = \alpha Y_i + \beta R(Y_i) - b = 0$$

- Newton iteration

$$Y^{k+1} = Y^k + \delta Y^k$$

$$\frac{\partial F}{\partial Y} \delta Y^k = F(Y_i^k)$$

- Convergence test

$$\left\| F(Y_i^{k+1}) \right\|_2 < \varepsilon_{atol}$$

$$\left\| F(Y_i^{k+1}) \right\|_2 < \varepsilon_{rtol} \left\| F(Y_i^k) \right\|_2$$

$$\left\| \delta Y^{k+1} \right\|_2 < \varepsilon_{stol}$$

Solution of implicit equations

- LU-SGS – linearization w.r.t. Q^n

Jameson and Yoon, 1987, *AIAAJ*
Yoon and Jameson, 1988, *AIAAJ*

$$\left(\alpha + \beta \frac{\partial R}{\partial q}\right) \delta q^n = b$$

$$L = \alpha I + \beta (D_x^- A^+ - A^- + \cdots)$$

$$L D^{-1} U \delta q^n = b \qquad D = \alpha I + \beta (A^+ - A^- + \cdots)$$

$$U = \alpha I + \beta (D_x^+ A^- + A^+ \ldots)$$

Dispersion and dissipation errors

- A-stable is NOT sufficient
  - Dispersion and dissipation errors
  - Oscillations of intermediate solutions
  - Numerical instability, increased CPU time

Dissipation terms

Model equation $$\frac{dQ}{dt} = \mu + i\lambda$$

Amplification Numerical $$G = \frac{Q^{n+1}}{Q^n}$$

Amplification Exact $$G_{exact} = e^{(\mu + i\lambda)\Delta t}$$

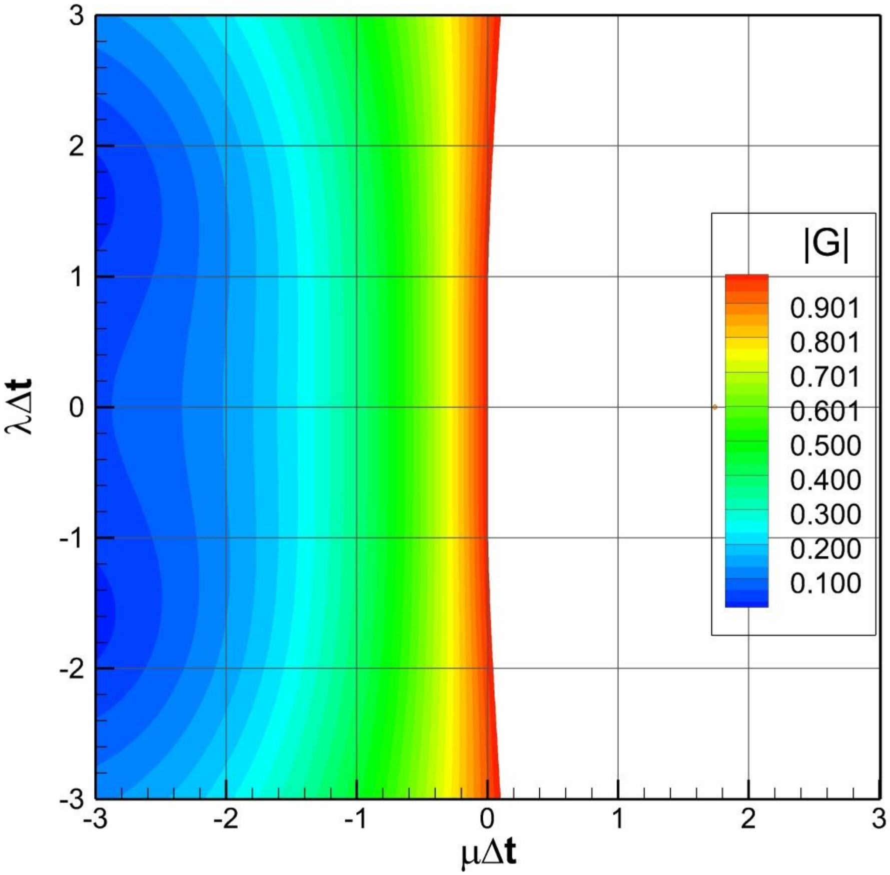
Error $$E = \frac{G}{G_{exact}}$$

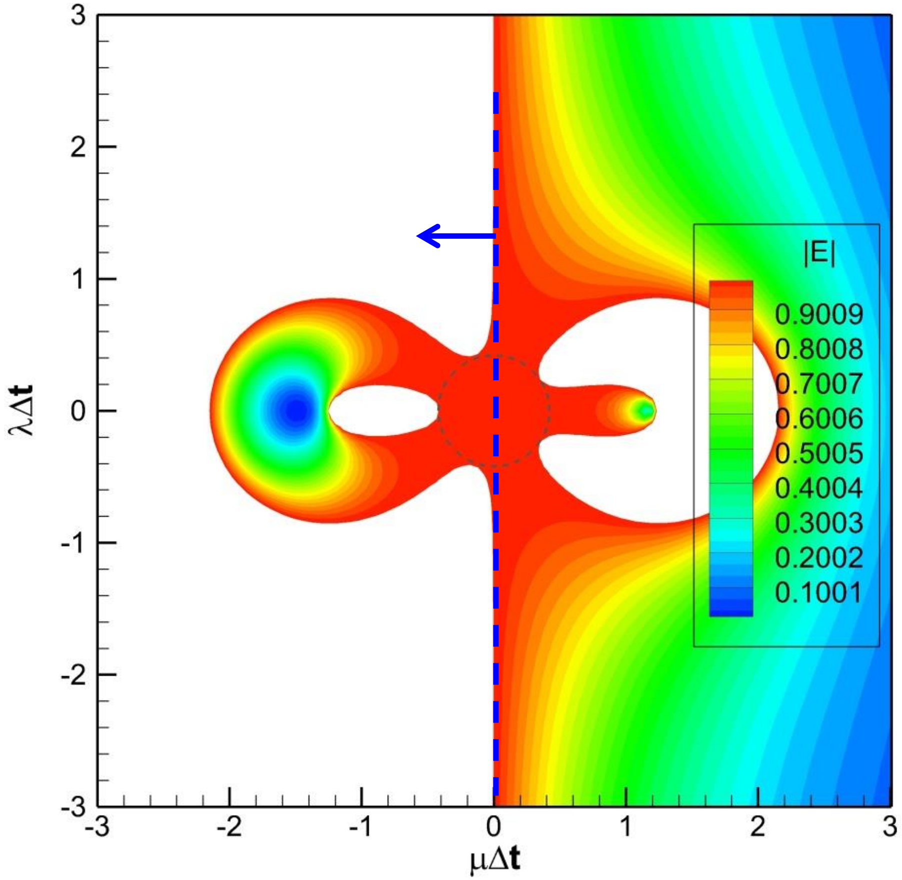
Dispersion and dissipation errors

Narazi et al., JCP, 2015

3-stage， 4^{th} order, A-stable

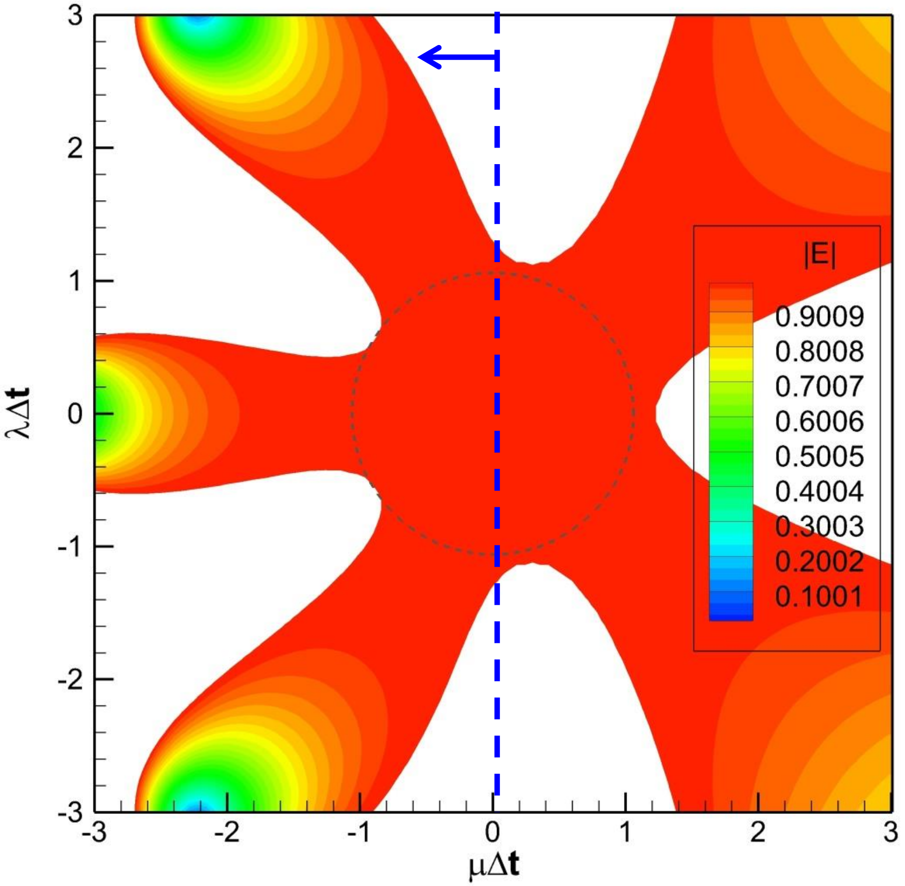
Unstable for CFL > 0.4 for multiple test cases

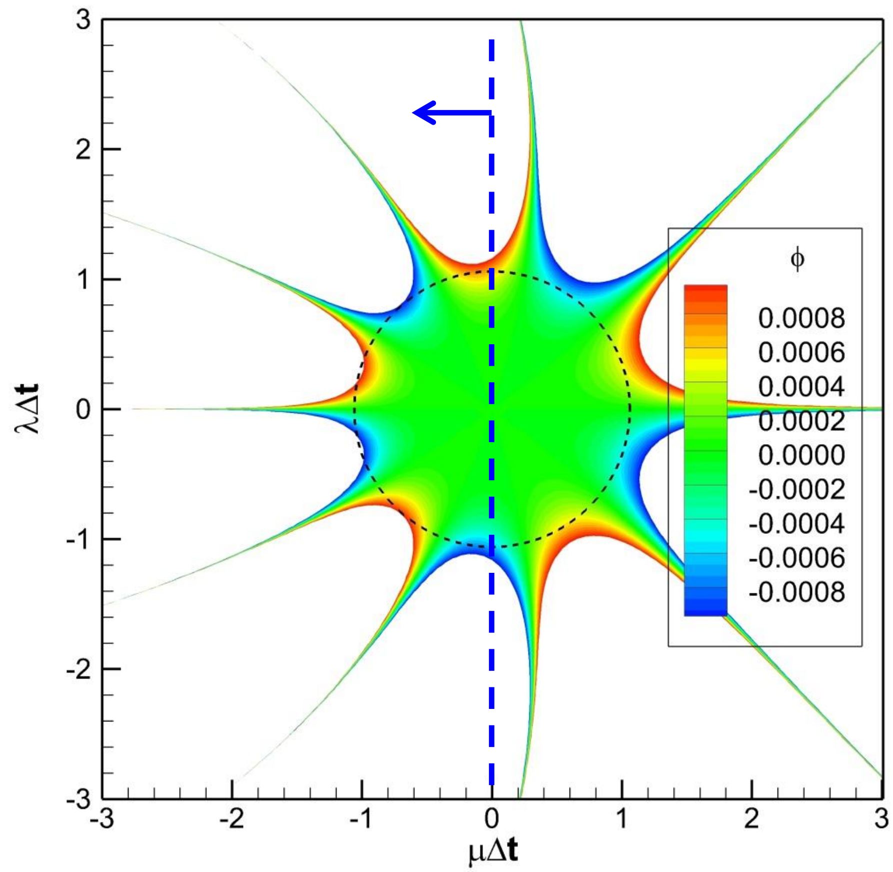




Dispersion and dissipation errors

Du & Ekaterinaris 2016
4^{th} order, not A-stable





Low-dispersion low-dissipation IMRK

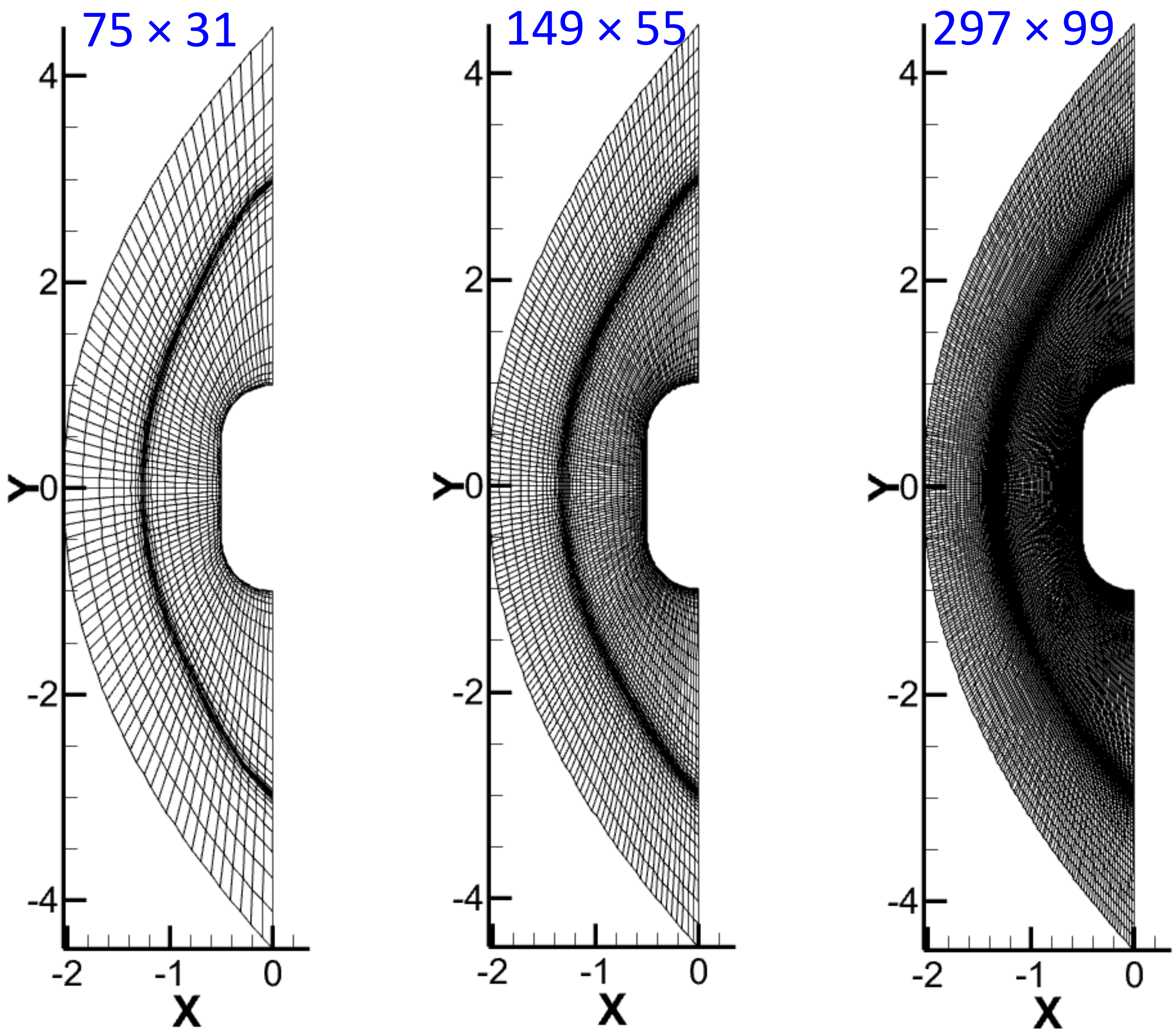
Du & Ekaterinaris (under review), 2016
4^{th} order, not A-stable

0.135497354012	0.135497354012			
0.510315342452	0.388552068308	0.121763274144		
0.870923224178	0.271286420495	0.470560027861	0.129076775822	
	0.316195414959	0.383754607157	0.300049977884	

Inviscid bow shock

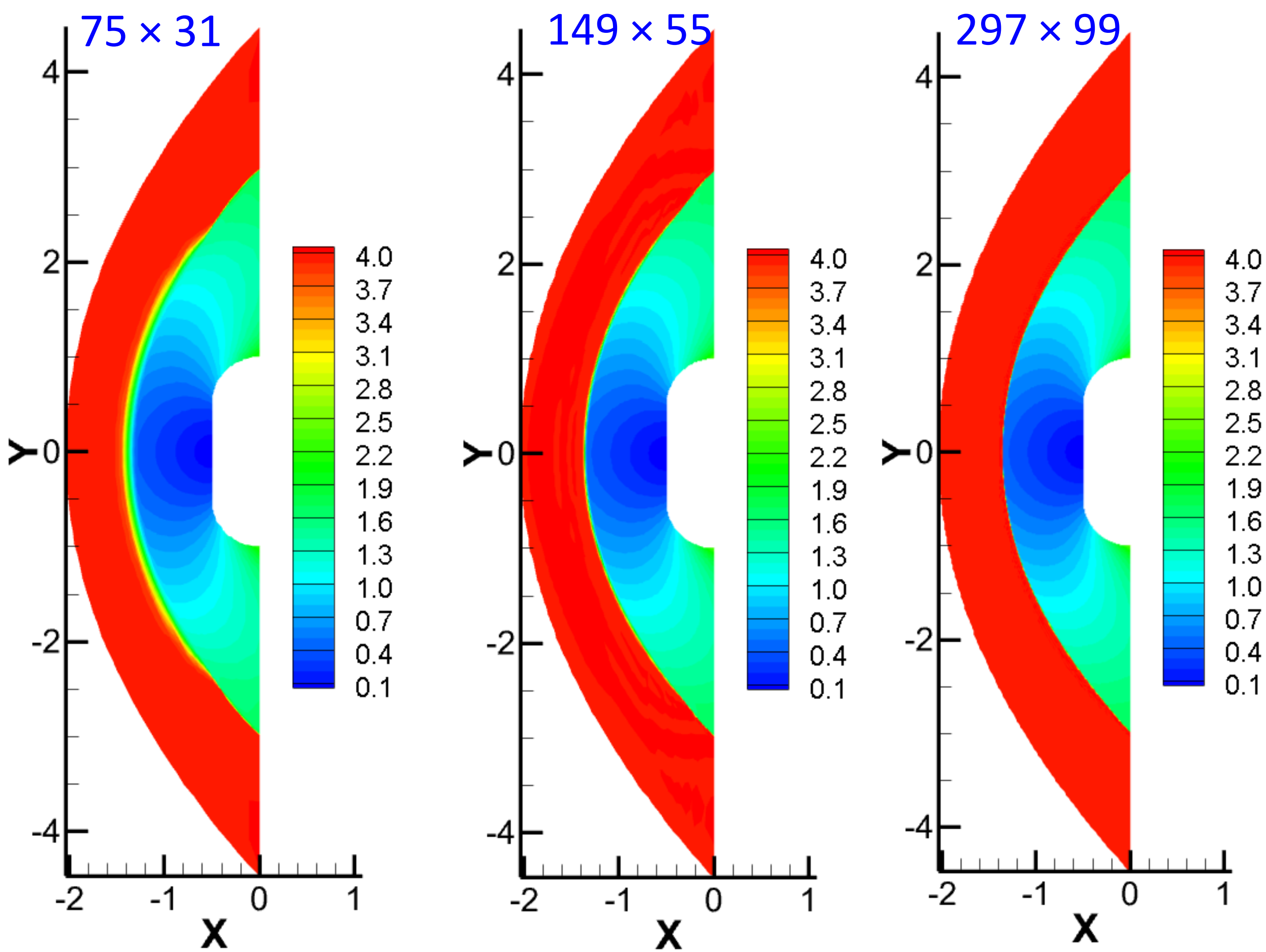
- Computational grids

$Ma_\infty = 4.0$



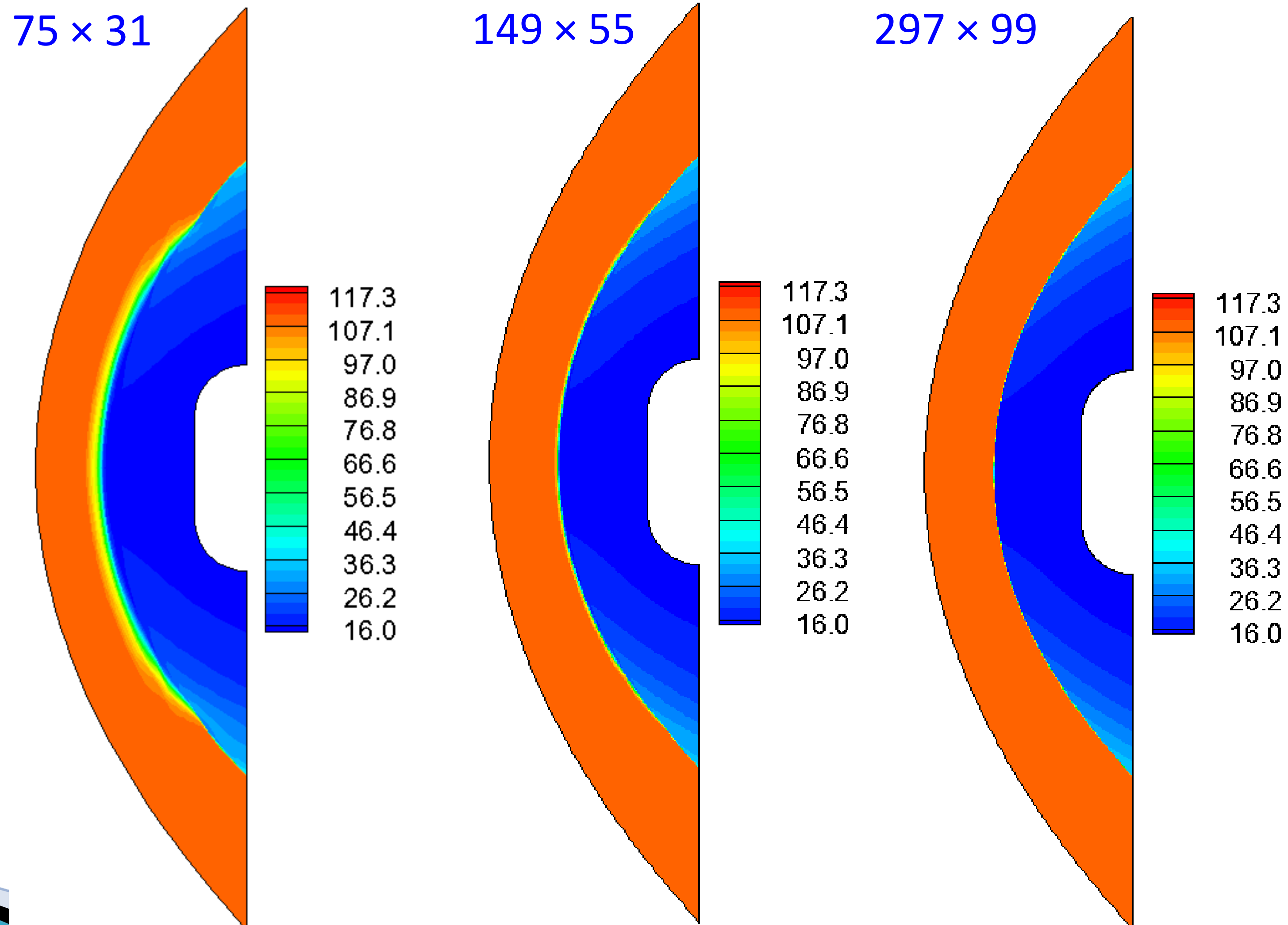
Inviscid bow shock

- Mach number contours



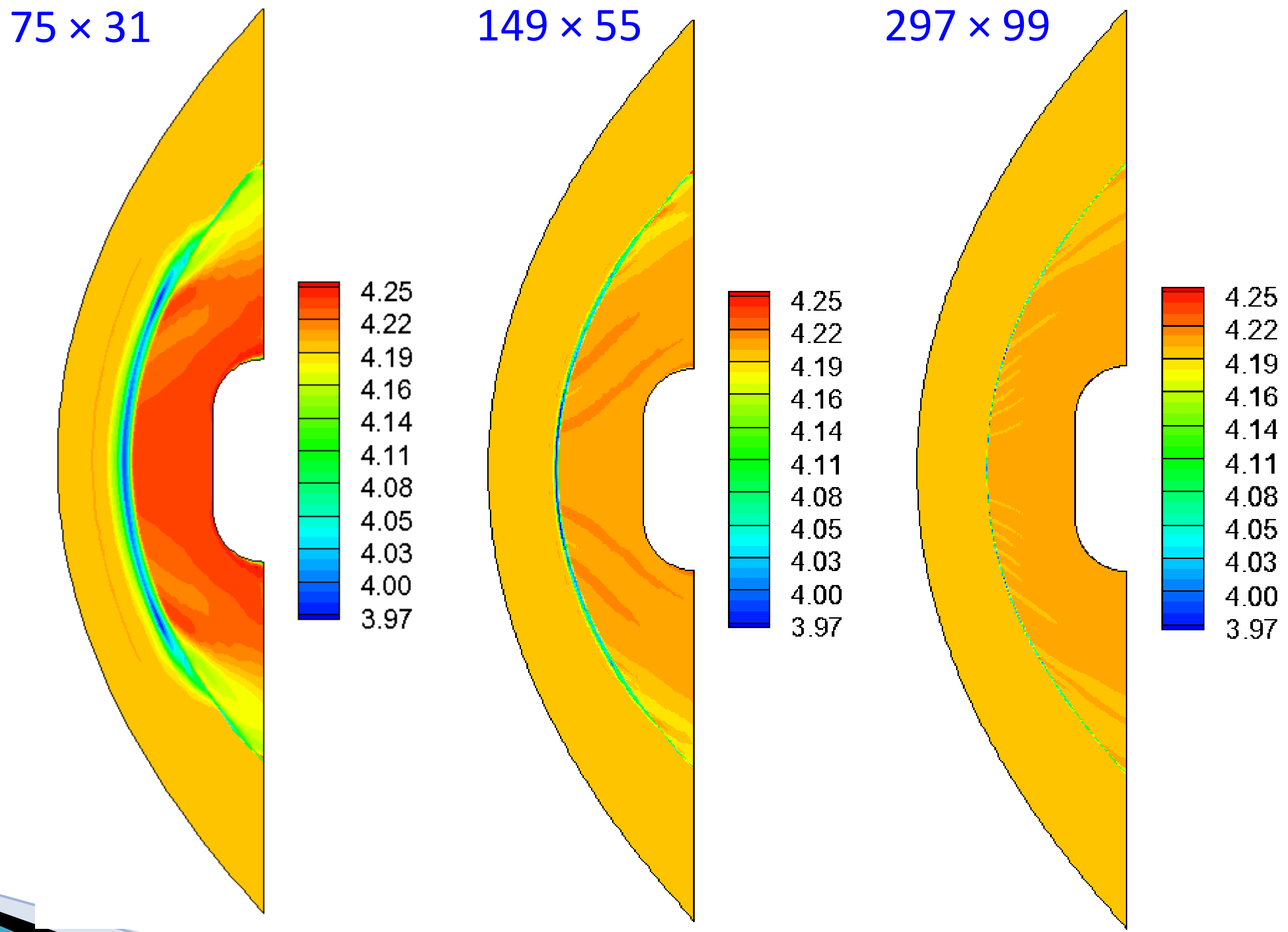
Inviscid bow shock

- Total pressure



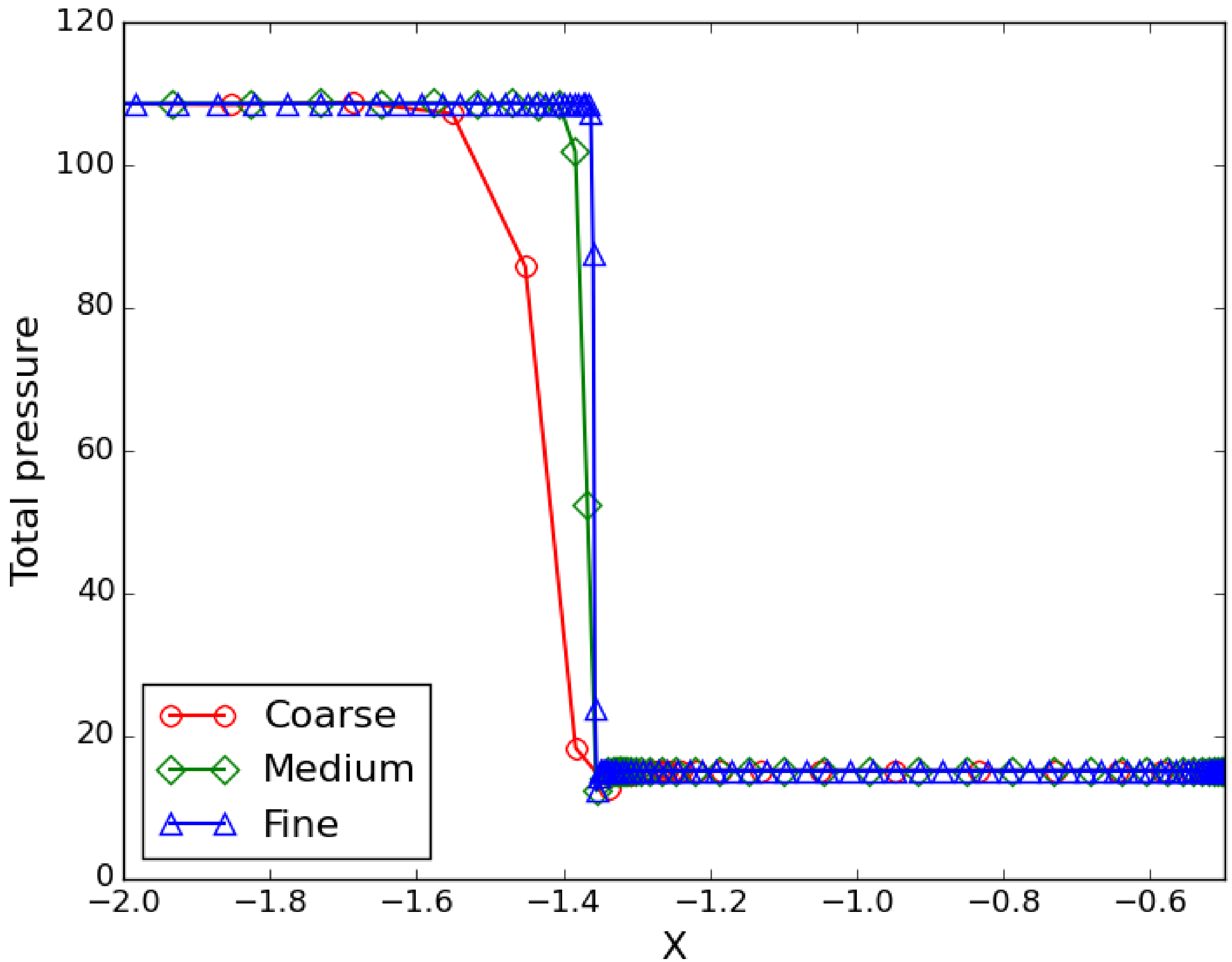
Inviscid bow shock

- Total temperature



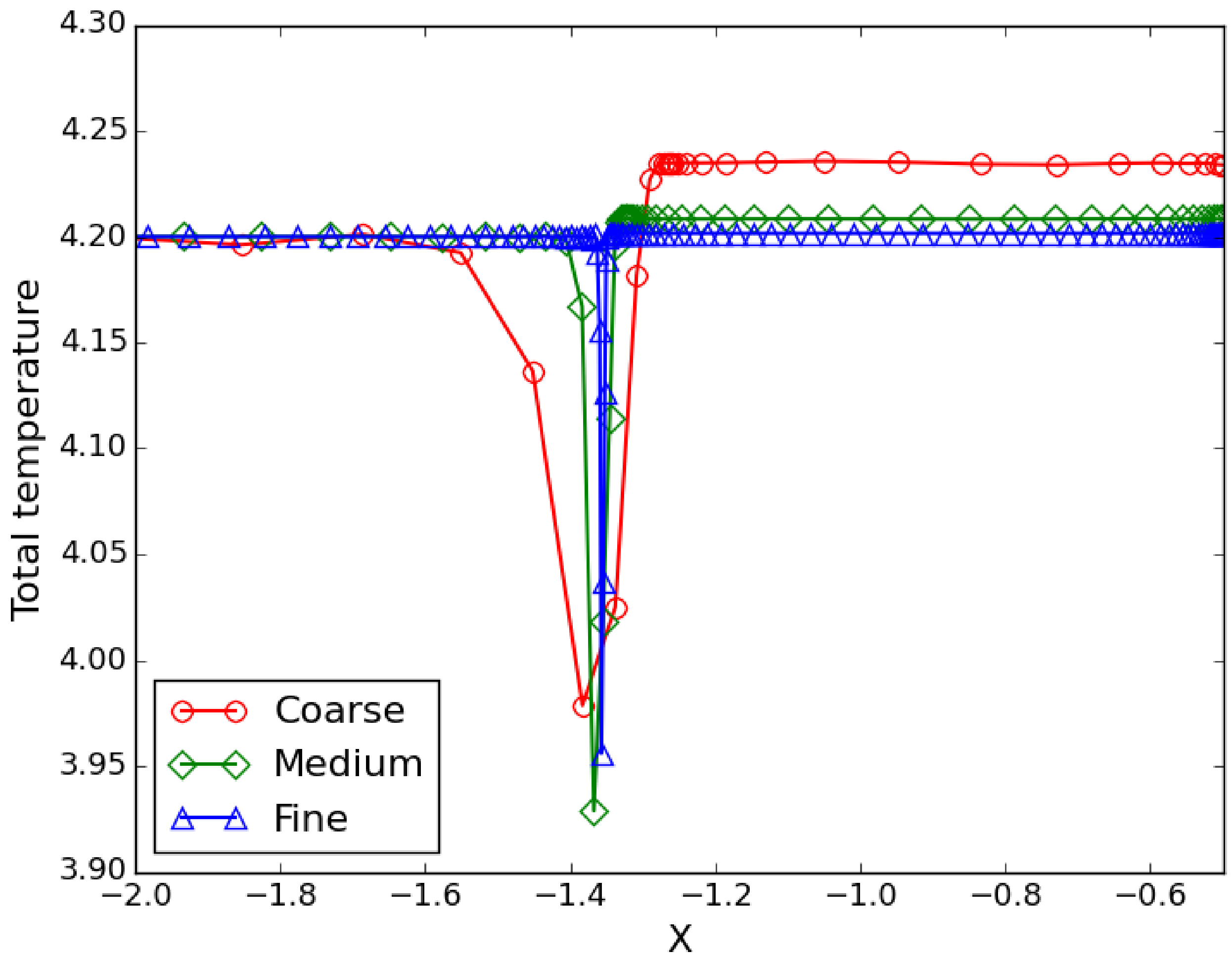
Inviscid bow shock

- Total pressure – symmetry line, y=0



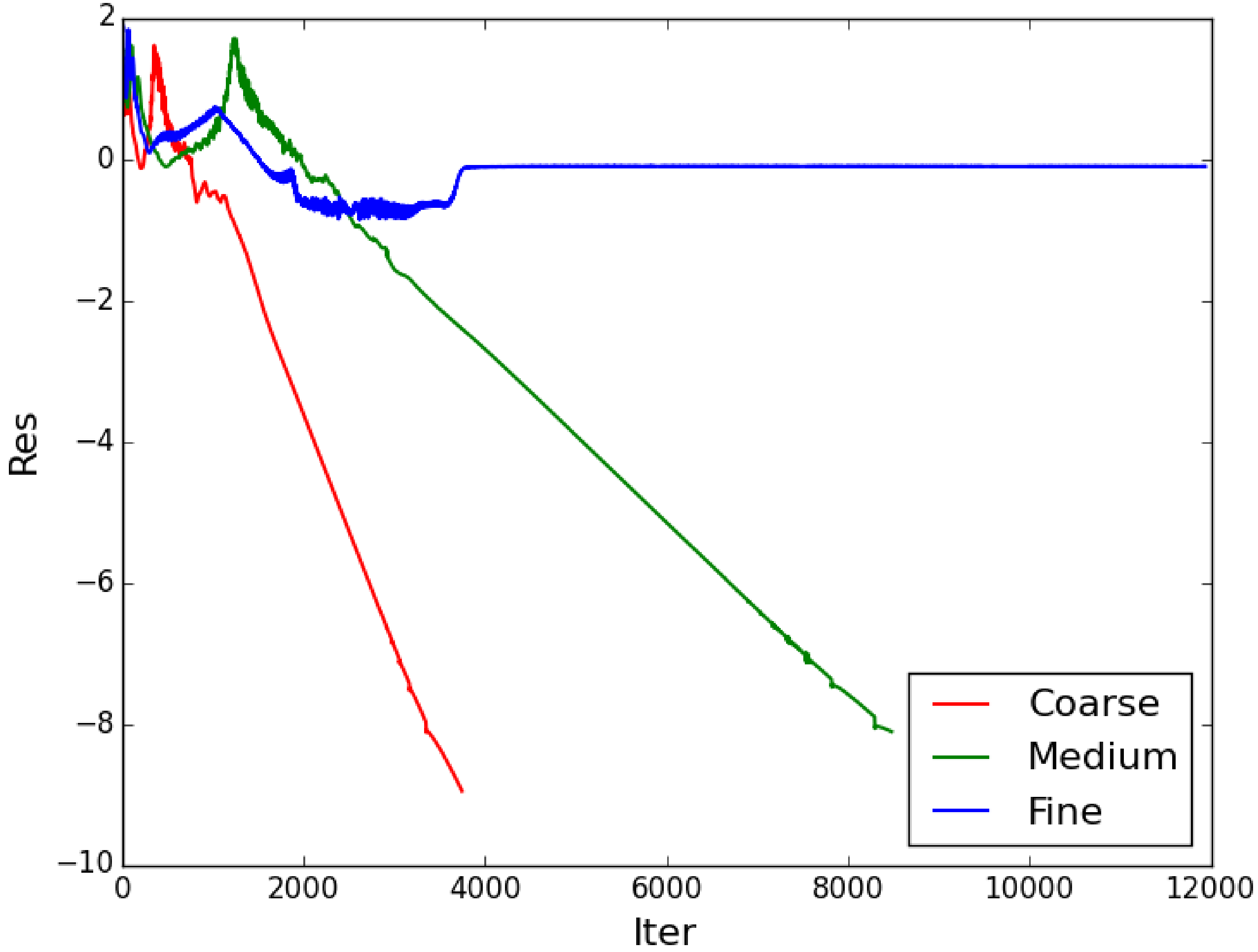
Inviscid bow shock

- Total temperature – symmetry line, y=0



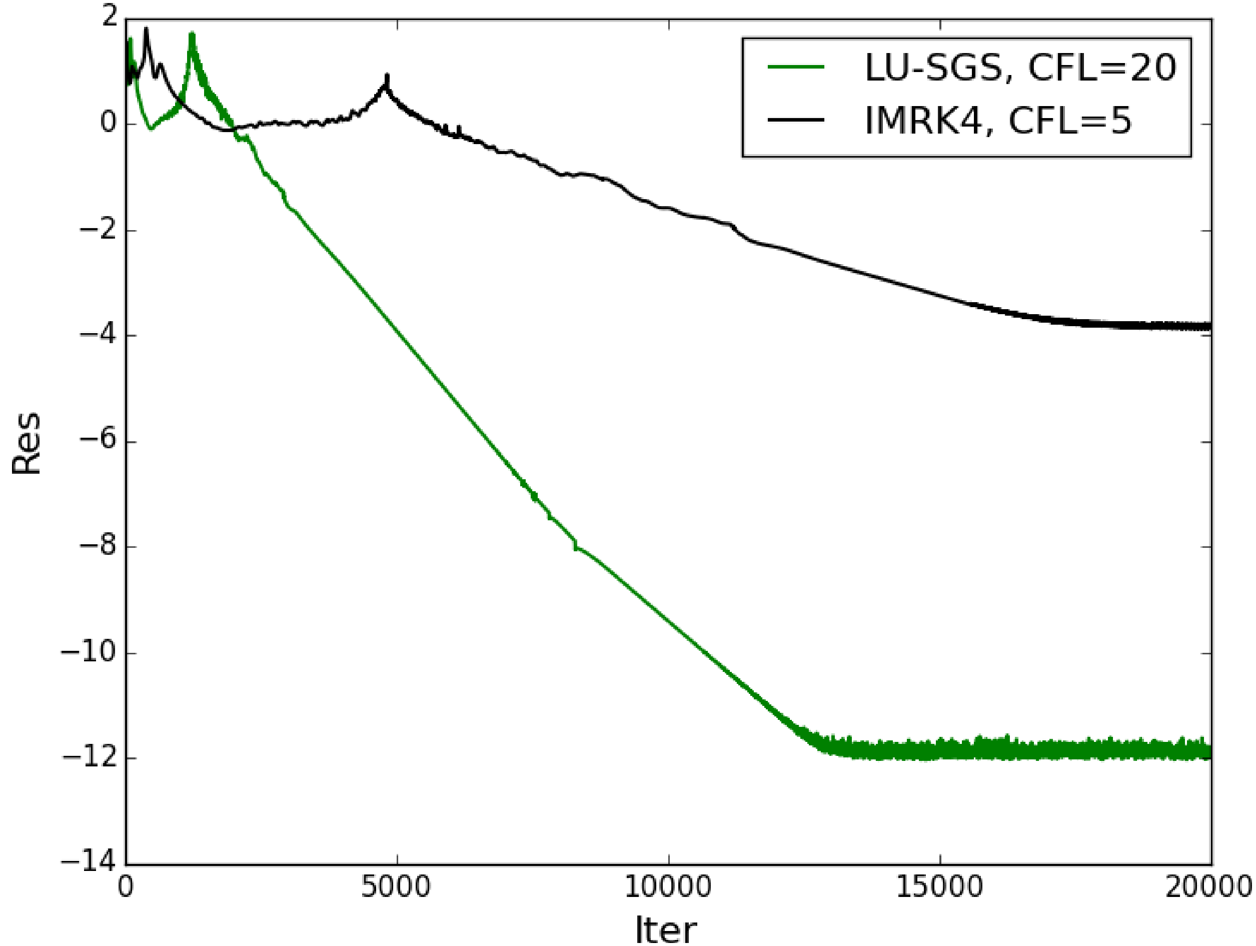
Convergence histories

- LU-SGS, CFL=20, continuity equation



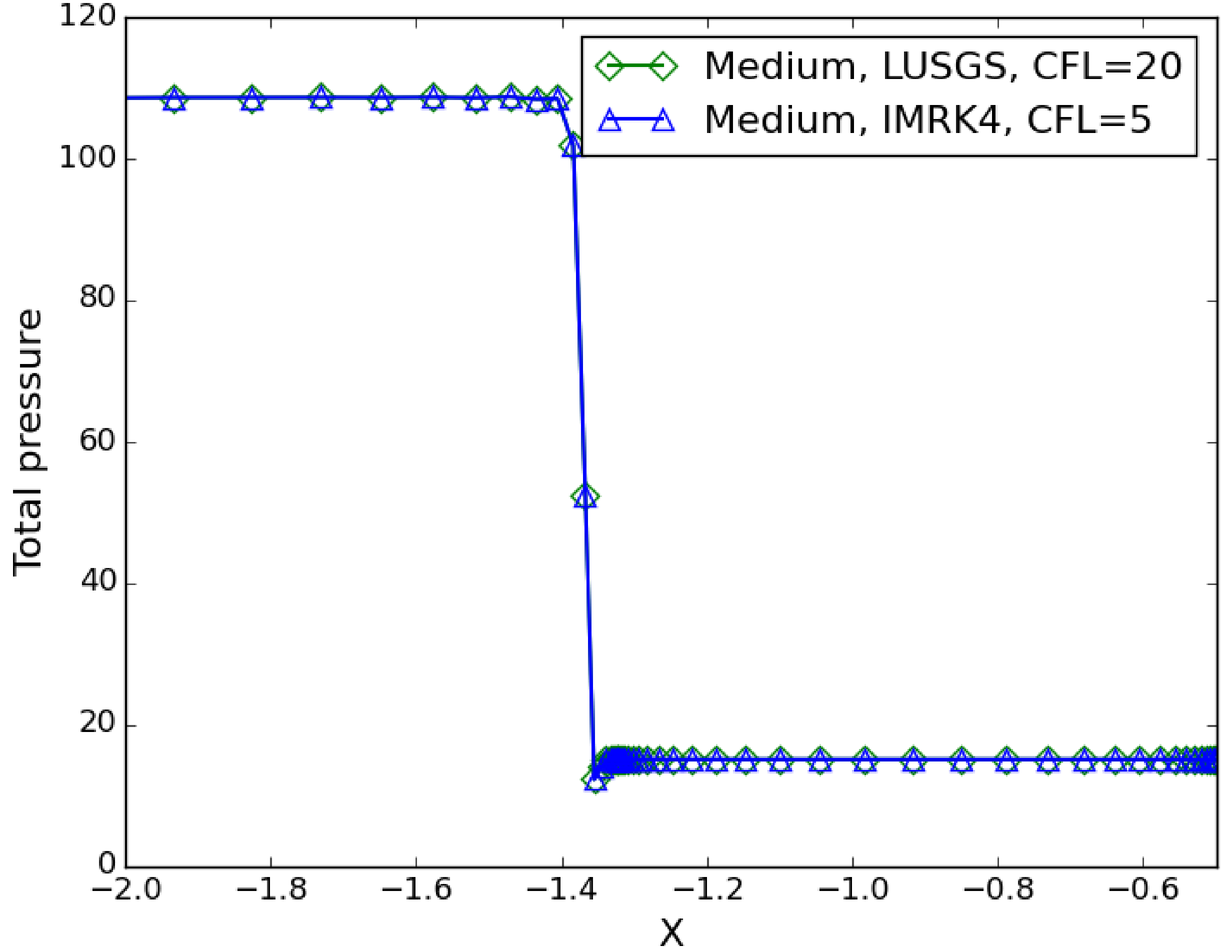
LU-SGS vs IMRK4

- Convergence histories, continuity equation



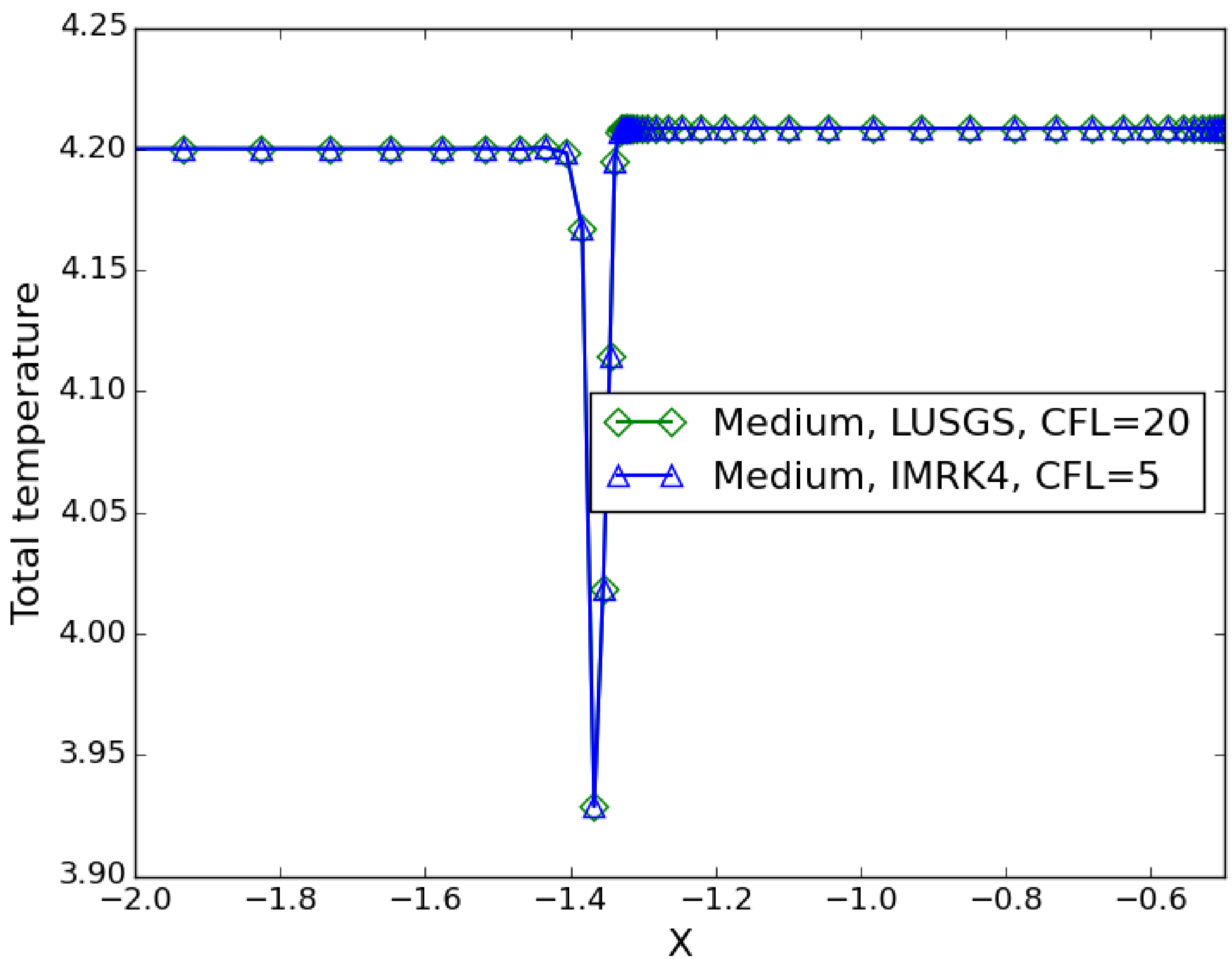
LU-SGS vs IMRK4

- Total pressure



LU-SGS vs IMRK4

- Total temperature



Summary

- The classical 5th order WENO scheme was applied for the computation of a bow shock at M = 4.

- Implicit marching was used for fast convergence to the steady state

- The numerical predictions for all meshes were in fair agreement

- A small amplitude jump of total temperature at the shock was found for all meshes

- A the total temperature before and after the shock remained almost constant for the fine mesh

- A small increase in total temperature was found with coarser meshes